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Dynamical Theory for Treating the Motion of Cold and Warm Fronts in the Atmosphere

by J. J. STOKER

Dynamics of Meteorological Fronts

by G. B. WHITHAM

CONTRACT N6ori-201, TASK ORDER NO. 1

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Dynamical Theory for Treating the Motion of Cold and Warm Fronts in the Atmosphere

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Sec. 1. Introduction

One of the most characteristic features of the motion of the atmosphere in middle latitudes and also one which is of basic importance in determining the weather there is the motion of wavelike disturbances which propagate on a discontinuity surface between a thin wedge-shaped layer of cold air on the ground and an overlying layer of warmer air. In addition to a temperature discontinuity there is also in general a discontinuity in the tangential component of the wind velocity in the two layers. The study of such phenomena was initiated long ago by Bjerknes and Solberg [2](*) and has been continued since by many others. In considering wave motions on discontinuity surfaces it was natural to begin by considering motions which depart so little from some constant steady motion (in which the discontinuity surface remains fixed in space) that linearizations can be performed, thus bringing the problems into the realm of the classical linear mathematical physics. Such studies have led to valuable insights, particularly with respect to the question of stability of wave motions in relation to the wave length of the perturbations. (The problems being linear, the motions in general can be built up as a combination, roughly speaking, of simple sine or cosine waves and it is the wave length of such components that is meant here, cf. Haurwitz [5], p. 234.) One conjecture is that the cyclones of the middle latitudes--probably the most striking single meteorological phenomenon there--are initiated because

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of the occurrence of such unstable waves on a discontinuity surface.

A glance at a weather map, or, still better, an examination of weather maps over a period of a few days, shows clearly that the wave motions on the discontinuity surfaces (which manifest themselves as the so-called fronts on the ground) develop amplitudes so rapidly and of such a magnitude that a description of the wave motions over a period of, say, a day by a linearization seems not feasible with any accuracy. The object of the present report is to make a first step in the direction of a nonlinear theory, based on the exact hydrodynamical equations, for the description of these motions that can be attacked by numerical or other methods. The reason that no such treatment seems to have been made up to now is not that meteorologists have failed to see the desirability of doing so, but rather that the difficulties to be overcome in obtaining even numerical solutions are really formidable. No claim is made that the problem is solved here in any general sense. What is done is to start with the general hydrodynamical equations and make a series of assumptions regarding the flow; in this way a sequence of three nonlinear problems (we call them Problems I, II, III), each one furnishing a consistent and complete mathematical problem, is formulated. In this way one can see the effect of each additional assumption in simplifying the mathematical problem. The first two problems result from a series of assumptions which would probably be generally accepted by meteorologists as reasonable, but unfortunately even Problem II is still unmanageable in terms of numerical analysis. Further, and more drastic, assumptions lead to a still simpler Problem III which is formulated in terms of three first order partial differential equations in three dependent and three independent variables (as contrasted with eight differential equations in four independent variables in Problem I). The three differential equations of Problem III are probably capable of yielding reasonably accurate approximations to the frontal motions under consideration, but they are still too difficult to deal with, even numerically, principally because they involve three independent variables(*)--such equations are well known to be beyond the scope of even the most modern digital computing machines. Consequently, still further simplifying assumptions must be made in order to obtain a theory capable of yielding some concrete results through calculation.

At this point, two different approaches to the problem have been devised. One of them, by Whitham (and which is discussed by him in this report), deals rather directly with the three differential equations of Problem III. Two of these equations are essentially the same as the classical gas dynamics equations in two independent and two dependent variables if one regards the third independent variable as a parameter and makes one additional and rather reasonable assumption. These two equations--which refer to motions in vertical planes--can therefore be integrated. Afterwards the transverse component of the velocity is found by integrating a linear first order partial differential equation. In this way a quite reasonable qualitative description of the dynamics of frontal motions can be achieved which is in good agreement with many of the observed phenomena. However, this theory has a disadvantage in that it does not

(*) The work of Freeman [3,4] is based on a theory which could be considered as a special case of Problem III in which the Coriolis terms are neglected and the motion is assumed at the outset to depend on only one space variable and the time. The idea of deriving the theory resulting in Problem III occurred to the author while reading Freeman's paper, and, indeed, Freeman indicates the possibility of doing so. The guiding idea is to make use of the analogue of the long-wave shallow water theory for gravity waves, and this idea seems to have occurred to a number of meteorologists (in particular, see Abdullah [1]). However, the present paper seems to be the first in which essentially three-dimensional motions are studied.

permit a complete numerical integration because of a peculiar difficulty at cold fronts. (The difficulty stems from the fact that a cold front corresponds in this theory to what amounts to the propagation of a shock into a vacuum--a mathematical impossibility. If one had a means of taking care of turbulence and friction at the ground, it would perhaps be possible to overcome this difficulty.) Nevertheless, the good qualitative agreement with the observed phenomena is an indication that the three differential equations furnishing the basic approximate theory from which we start--i.e. those of our Problem III--have in them the possibility of furnishing reasonable results.

The author's method of treating the three basic differential equations is quite different from that of Whitham, but it unfortunately involves a further assumption which has the effect of limiting the applicability of the theory. The guiding principle was that differential equations in only two independent variables should be obtained in order to make numerical computations feasible. On the other hand, the number of dependent variables need not be so ruthlessly limited. Finally, it is highly desirable to obtain differential equations of hyperbolic type in order that the method of characteristics can be employed as an aid in computing solutions. These objectives can be attained by making quite a few further simplifying assumptions with respect to the mechanics of the situation. The result is what might be called Problem IV. What remains of the original hydrodynamical equations is still complicated enough, however, to make one feel that if such a simplified theory fails to mirror the observed facts at least roughly, then the problem of calculating the unsteady motion of cold and warm fronts will present a formidable challenge to numerical analysis in the general field of nonlinear partial differential equations. The theory formulated in Problem IV is embodied in a system of four nonlinear first order partial differential equations of hyperbolic type in four dependent and two independent variables. A numerical integration of these equations is being carried out (by finite differences applied to the equations in characteristic form), but the labor of integrating the equations is so great that only meagre results are so far available.

In section 2 the sequence of approximate theories formulated as Problems I, II, III will be derived from the hydrodynamical equations by adding successively more and more hypotheses of a physical character. In section 3 the author's method of treating Problem III--which should be regarded as the basic problem--approximately will be derived. This yields Problem IV. In section 4 the characteristic form of the differential equations of Problem IV will be obtained and conclusions drawn from them. This is followed by the report of G. Whitham giving his method of treatment of the differential equations of the basic Problem III. The theory of G. Whitham is in some ways more satisfactory than that of the author since it yields important qualitative results of the kind actually observed, and does that, in addition, without the necessity to make numerical calculations. However, it is unsuitable for describing the early stages of frontal motions, and this suggests that the method of the author might be used for the initial stages followed by the method of Whitham for the later stages of the motions.

Sec. 2. Derivation of the basic approximate theory

In the present section the details of the derivation of the basic approximate theory will be given, but it seems worth while to sketch out in general terms the underlying ideas and motivations first by way of orientation.

To begin with, a certain steady motion (called a stationary front) is taken as an initial state, and this consists of a uniform flow of two superimposed layers of cold and warm air, as indicated in Figure 1. The z-axis is taken positive upward and the x,y-plane is a tangent plane to the earth. The rotation of the earth is to be taken into account

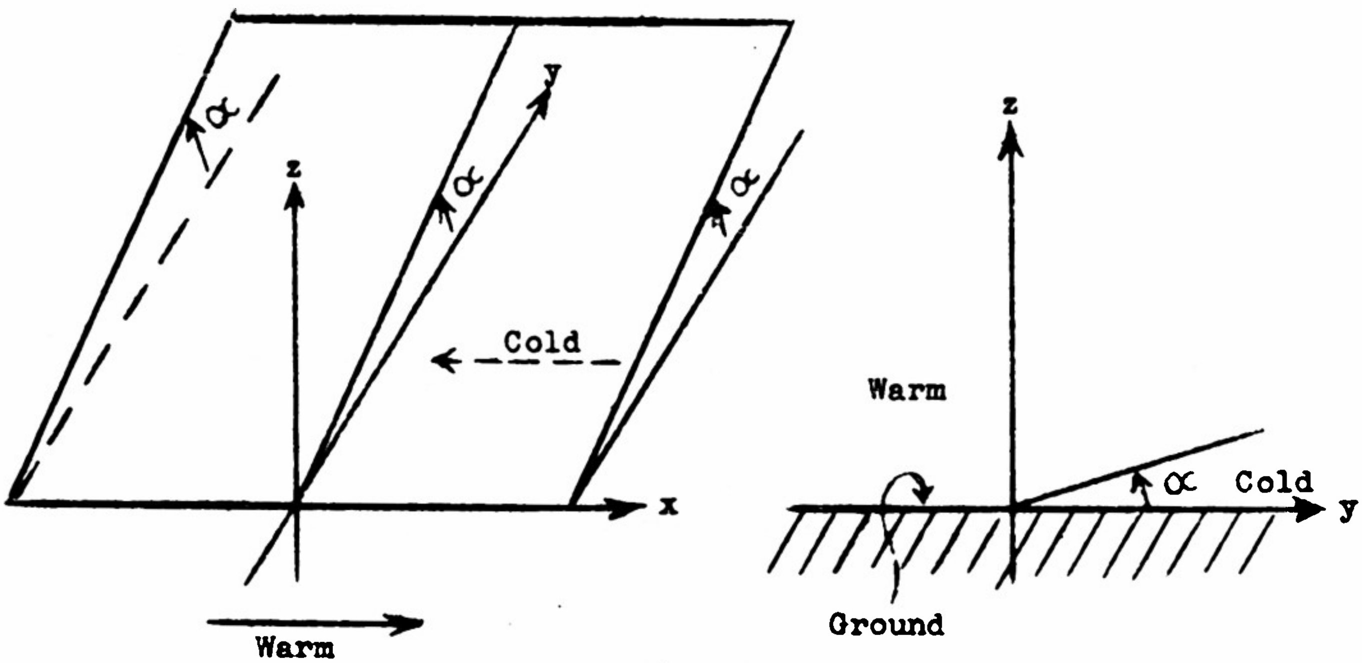

Fig. 1
A stationary front

but, for the sake of simplicity, not its sphericity--a common practice in dynamic meteorology. The coordinate system is assumed to be rotating about the z-axis with a constant angular velocity $\Omega = \omega \sin \varphi$, with ω the angular velocity of the earth and φ the latitude of the origin of our coordinate system. (The motivation for this is that the main effects one cares about are found so long as the Coriolis term is included, and that neglect of the curvature of the earth has no serious qualitative effect.) As indicated in Figure 1, the cold air lies in a wedge under the warm air and the discontinuity surface between the two layers is inclined at angle α to the horizontal. The term "front" is always applied to the intersection of the discontinuity surface with the ground, and in the present case we have therefore as initial state a stationary front running along the x-axis. The wind velocity in the two layers is parallel to the x-axis (otherwise the discontinuity surface could not be stationary), but it will in general be different in magnitude and perhaps even opposite in direction in the two layers. The situation shown in Figure 1 is not uncommon. For instance, the x-axis might be in the eastward direction, the y-axis in the northward direction and the warm air would be moving in the direction of the prevailing westerlies. The origin of the cold air at the ground is, of course, the existence of the cold polar regions. We shall see later that such configurations are dynamically correct and that the angle α is uniquely determined (and quite small, of the order of $\frac{1}{2}^{\circ}$) once the state of the warm air and cold air is given. (The discontinuity surface is not horizontal because of the Coriolis force arising from the rotation of the earth.)

We proceed next to describe what is observed to happen in many cases once such a stationary front starts moving. In Figure 2 (see next page) a sequence of diagrammatic sketches is given which indicate in a general way what can happen. All of the sketches show the intersection of the moving discontinuity surface (cf. Figure 1) with the ground (the x,y-plane with the y-axis taken northward, the x-axis taken eastward). The shaded area indicates the region on the ground covered by cold air, while the unshaded region is covered at the ground by warm air. Of course, the cold air always lies in a thin wedge under a thick layer of warm air. In Figure 2a the development of a bulge in the stationary front toward the north is indicated.(*) Such a bulge then

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(*) What agency serves to initiate and to maintain such motions appears to be a mystery. Naturally such an important matter has been the subject of a great deal of discussion and speculation, but there seems to be no consistent view about it among meteorologists. In applying the theory derived here no attempt is made to settle this question: we simply take our dynamical model, assume an initial condition which in effect states that a bulge of the kind just described is initiated, and then study the subsequent motion by integrating the differential equations subject to the appropriate initial conditions. However, if the approximate theory is really valid, such studies might perhaps be used, or could be modified, in such a way as to throw some light on this important and vexing question.

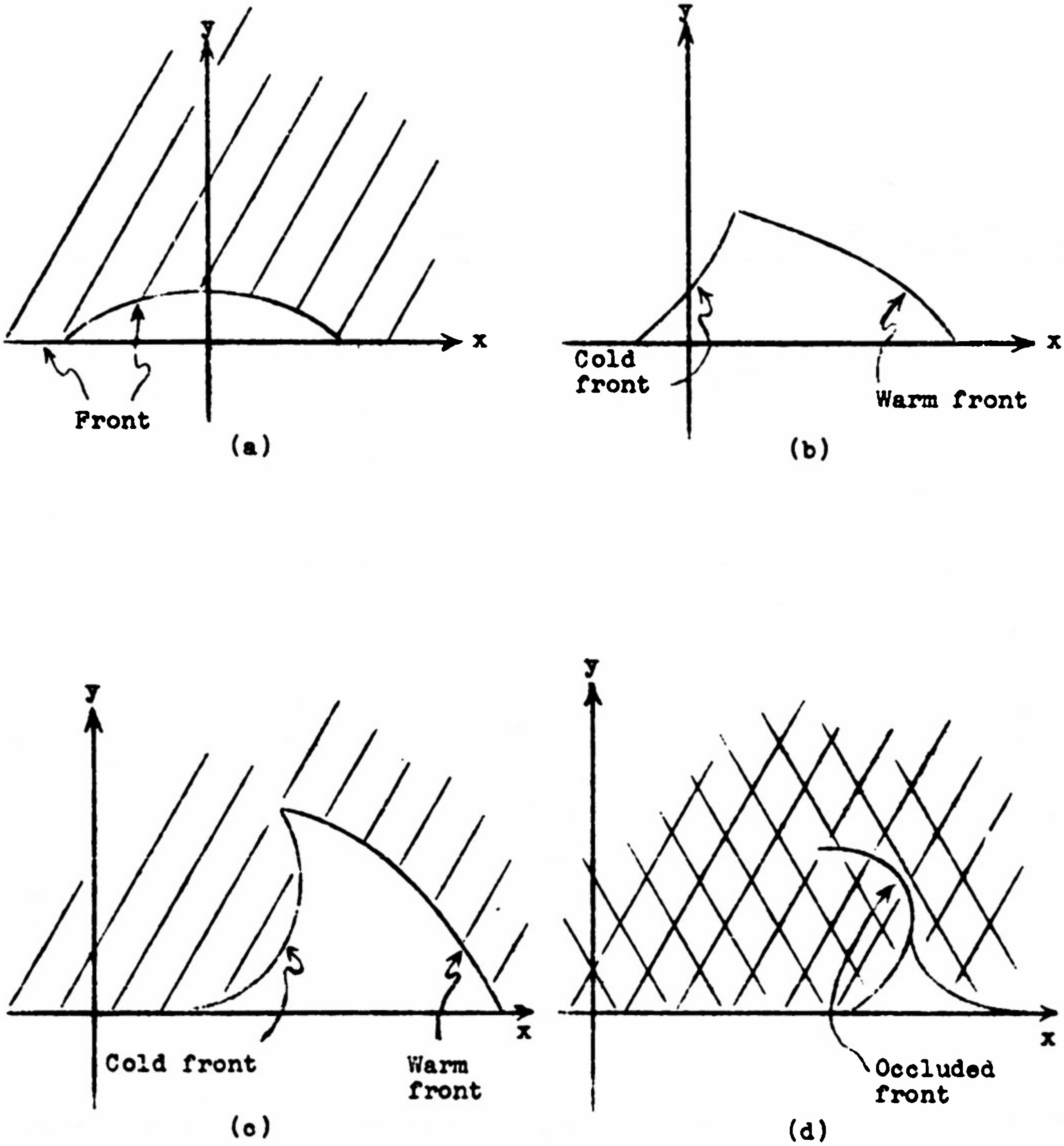

Fig. 2
Stages in the motion of a frontal disturbance

deepens and propagates eastward with a velocity of the order of 500 miles per day. It now becomes possible to define the terms <u>cold front</u> and <u>warm front</u>. As indicated in Figure 2b, the cold front is that part of the whole front at which cold air is taking the place of warm air at the ground, and the warm front is the portion of the whole front where cold air is retreating with warm air taking its place at the ground. Since such cold and warm fronts are accompanied by winds, and by precipitation in various forms--in fact, by all of the ingredients that go to make up what one calls weather--it follows that the weather at a given locality in the middle latitudes is largely conditioned by the passage of such frontal disturbances. Cold fronts and warm fronts behave differently in many ways. For example, the cold front in general moves faster than the warm front and steepens relative to it, so that an originally symmetrical disturbance or wave gradually becomes distorted in the manner indicated in Figure 2c. This process sometimes--though by no means always--continues until the greater portion of the cold front has overrun the warm front; an occluded front, as indicated in Figure 2d, is then said to occur. The prime object of what follows is to derive a theory--or perhaps better, to invent a simplified dynamical model--capable of dealing with fluid motions of this type that is not on the one hand so crude as to fail to yield at least roughly the observed motions, and on the other hand is not impossibly difficult to use for the purpose of mathematical discussion and numerical calculation.

We begin with the classical hydrodynamical equations. The equations of motion in the so-called Eulerian form are taken:

$$
(2.1) \qquad \begin{cases} \rho \dfrac{Du}{Dt} = -\dfrac{\partial p}{\partial x} + \rho F_{(x)} \\ \rho \dfrac{Dv}{Dt} = -\dfrac{\partial p}{\partial y} + \rho F_{(y)} \\ \rho \dfrac{Dw}{Dt} = -\dfrac{\partial p}{\partial z} + \rho F_{(z)} - \rho g \end{cases}
$$

with $\frac{D}{Dt}$ (the particle derivative) defined by the operator $\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$. In these equations u, v, w are the velocity components relative to our rotating coordinate system, p is the pressure, ρ the density, $\rho F_{(x)}$ etc. the components of the Coriolis force due to the rotation of the coordinate system, and ρg is the force of gravity (assumed to be constant). These equations hold in both the warm air and the cold air, but it is preferable to distinguish the dependent quantities in the two different layers; this is done here throughout by writing u', v', w' for the velocity components in the warm air and similarly for the other dependent quantities.

We now introduce an assumption which is commonly made in dynamic meteorology in discussing large-scale motions of the atmosphere, i.e. that the air is incompressible. In spite of the fact that such an assumption rules out thermodynamic processes, it does seem rather reasonable since the pressure gradients which operate to create the flows of interest to us are quite small and, what is perhaps the decisive point, the propagation speed of the disturbances to be studied is very small compared with the speed of sound in air (i.e. with disturbances governed by compressibility effects). It would be possible to consider the atmosphere, though incompressible, to be of variable density. However, for the purpose of obtaining as simple a dynamical model as possible it seems reasonable to begin with an atmosphere having a constant density in each of the two layers. As a consequence of these assumptions we have the following equation of continuity:

$$u_x + v_y + w_z = 0. \tag{2.2}$$

The equations (2.1) and (2.2) together with the conditions of continuity of the pressure and of the normal velocity components on the discontinuity surface, the condition $w = 0$

at the ground, appropriate initial conditions, etc. doubtlessly yield a mathematical problem--call it Problem I--the solution of which would furnish reasonably good approximations to the observed phenomena. Unfortunately, such a problem is still so difficult as to be far beyond the scope of known methods of analysis--including analysis by numerical computation. Thus still further simplifications are in order.

One of the best-founded empirical laws in dynamic meteorology is the hydrostatic pressure law, which states that the pressure at any point in the atmosphere is very closely equal to the static weight of the column of air vertically above it. This is equivalent to saying that the acceleration terms in the third equation of (2.1) can be ignored, with the result

$$\frac{\partial p}{\partial z} = -\rho g. \tag{2.3}$$

This is also the basis of the so-called long-wave or shallow water theory of surface gravity waves, and the author was guided in much that follows by experience gained (cf. [6]) in working with such gravity wave problems. Since the vertical component of the acceleration of the particles is ignored, it follows on purely kinematical grounds that the horizontal components of the velocity will be independent of the vertical coordinate z for all time if that was the case at the initial instant $t = 0$. Since we do in fact assume an initial motion with such a property, it follows that we have

$$u = u(x,y,t), \qquad v = v(x,y,t), \qquad w = 0. \tag{2.4}$$

The first two of the equations of motion (2.1) and the equation of continuity (2.2) therefore reduce to

$$(2.5) \qquad \begin{cases} u_t + uu_x + vu_y = -\frac{1}{\rho} p_x + F_{(x)} \\ v_t + uv_x + vv_y = -\frac{1}{\rho} p_y + F_{(y)} \\ u_x + v_y = 0, \end{cases}$$

where we use subscripts to denote partial derivatives and subscripts enclosed in parentheses to indicate components of a vector. The Coriolis acceleration terms are now given by

$$(2.6) \qquad \begin{cases} F_{(x)} = 2\omega \sin \varphi \cdot v = \lambda v \\ F_{(y)} = -2\omega \sin \varphi \cdot u = -\lambda u \end{cases}$$

when use is again made of the fact that $w = 0$. (The latitude angle φ, as was indicated earlier, is assumed to be constant.) We observe once more that all of these relations hold in both the warm and cold layers, and we distinguish between the two when necessary by a prime on the symbols for quantities in the warm air. It is perhaps also worth mentioning that the equations (2.5) with $F_{(x)}$ and $F_{(y)}$ defined by (2.6) are valid for all orientations of the x,y-axes; thus it is not necessary to assume (as we did above, for example) that the original stationary front runs in the east-west direction.

We have not so far made full use of the hydrostatic pressure law (2.3). To this end it is useful to introduce the vertical height $h = h(x,y,t)$ of the discontinuity surface between the warm and cold layers and the height $h' = h'(x,y,t)$ of the warm layer itself (see Figure 3). Assuming that the pressure p' is zero at the top of the warm layer we find by integrating (2.3):

$$(2.7) \qquad p'(x,y,z,t) = \rho' g(h' - z)$$

for the pressure at any point in the warm air. In the cold

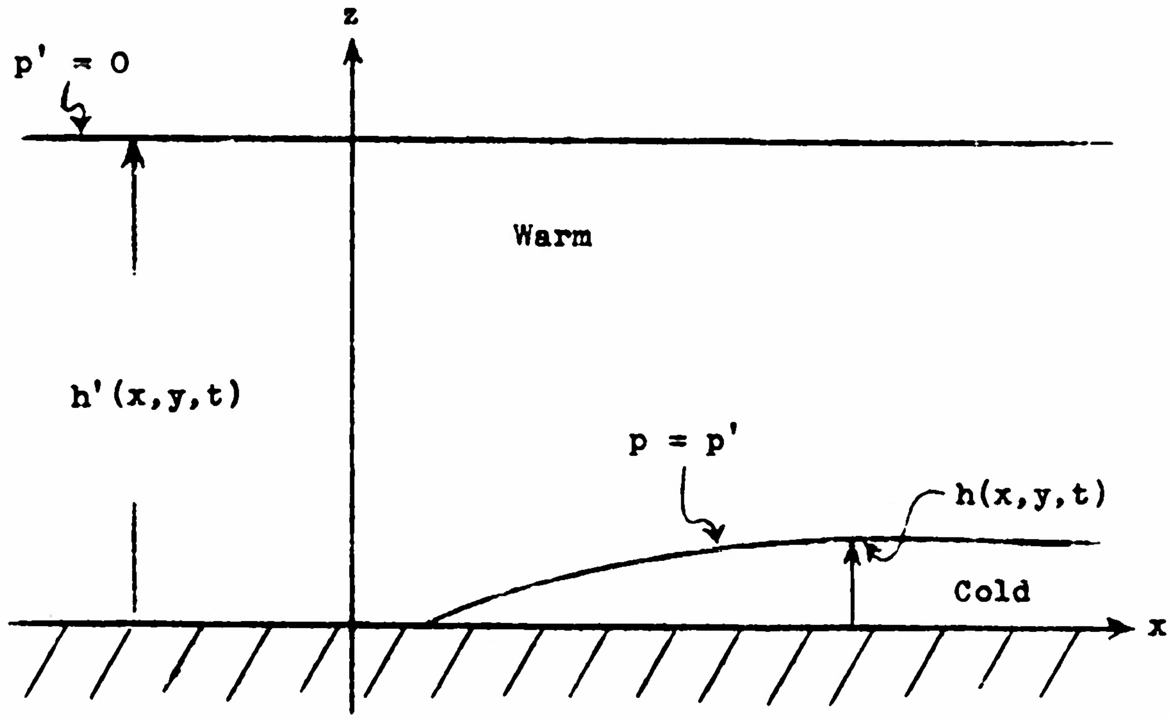

Fig. 3
Vertical height of the two layers

air we have, in similar fashion:

$$p(x,y,z,t) = \rho' g(h' - h) + \rho g(h - z) \tag{2.8}$$

when the condition of continuity of pressure, $p' = p$ for $z = h$, is used. (The formula (2.8) is the starting point of the paper by Freeman [3] which was mentioned in the introduction.) Insertion of (2.8) in (2.5) and of (2.7) in (2.5)' yields the following six equations for the six quantities u, v, h, u', v', h':

(2.9) (cold air)

$$\begin{cases} u_t + uu_x + vu_y = -g[\frac{\rho'}{\rho} h'_x + (1 - \frac{\rho'}{\rho})h_x] + \lambda v \\ v_t + uv_x + vv_y = -g[\frac{\rho'}{\rho} h'_y + (1 - \frac{\rho'}{\rho})h_y] - \lambda u \\ u_x + v_y = 0 \end{cases}$$

$$
(2.10)\ \text{(warm air)}\quad
\begin{cases}
u'_t + u'u'_x + v'u'_y = -gh'_x + \lambda v' \\
v'_t + u'v'_x + v'v'_y = -gh'_y - \lambda u' \\
u'_x + v'_y = 0.
\end{cases}
$$

These equations together with the kinematic conditions appropriate at the surfaces $z = h$ and $z = h'$, and initial conditions at $t = 0$ would again constitute a reasonable mathematical problem--call it Problem II--which could be used to study the dynamics of frontal motions. The problem II is much simpler than the problem I formulated above in that the number of dependent quantities is reduced from eight to six and, probably still more important, the number of independent variables is reduced from four to three. These simplifications, it should be noted, come about solely as a consequence of assuming the hydrostatic pressure law, and since meteorologists have much evidence supporting the validity of such an assumption, the problem II should then furnish a reasonable basis for discussing the problem of frontal motions. Unfortunately, Problem II is just about as inaccessible as Problem I from the point of view of mathematical and numerical analysis. Consequently, we make still further hypotheses leading to a simpler theory.

As a preliminary to the formulation of Problem III we write down the kinematic free surface conditions at $z = h$ and $z = h'$ (the dynamical free surface conditions, $p = 0$ at $z = h'$ and $p = p'$ at $z = h$, have already been used). These conditions state simply that the particle derivatives of the functions $z - h(x,y,t)$ and $z - h'(x,y,t)$ vanish, since any particle on the surface $z - h = 0$ or the surface $z - h' = 0$ remains on it. We have therefore the conditions

$$(2.11)\qquad \begin{cases} uh_x + vh_y + h_t = 0 \\ u'h_x + v'h_y + h_t = 0 \\ u'h'_x + v'h'_y + h'_t = 0, \end{cases}$$

in view of the fact that w vanishes everywhere. It is convenient to replace the third equations (the continuity equations) in the sets (2.9) and (2.10) by

$$(2.12)\qquad (uh)_x + (vh)_y + h_t = 0, \quad \text{and}$$

$$(2.13)\qquad [u'(h' - h)]_x + [v'(h' - h)]_y + (h' - h)_t = 0,$$

which are readily seen to hold because of (2.11). In fact, the last two equations simply state the continuity conditions for a vertical column of air extending (in the cold air) from the ground up to $z = h$, and (in the warm air) from $z = h$ to $z = h'$.

We now make the really trenchant assumption, i.e. that <u>the motion of the warm air layer is not affected by the motion of the cold air layer</u>. This assumption has a rather reasonable physical basis, as might be argued in the following way: Imagine the stationary front to have developed a bulge in the y-direction, say, as in Figure 4a (see next page). The warm air can adjust itself to the new condition simply through a slight change in its vertical component, without any need for a change in u' and v', the horizontal components. This is indicated in Figure 4b, which is a vertical section of the air taken along the line AB in Figure 4a; in this figure the cold layer is shown with a quite small height--which is what one always assumes. Since we ignore changes in the vertical velocity components in any case, it thus seems reasonable to make our assumption of unaltered flow conditions in the warm air. However, in the cold air one sees readily--as indicated in Figure 4c--that

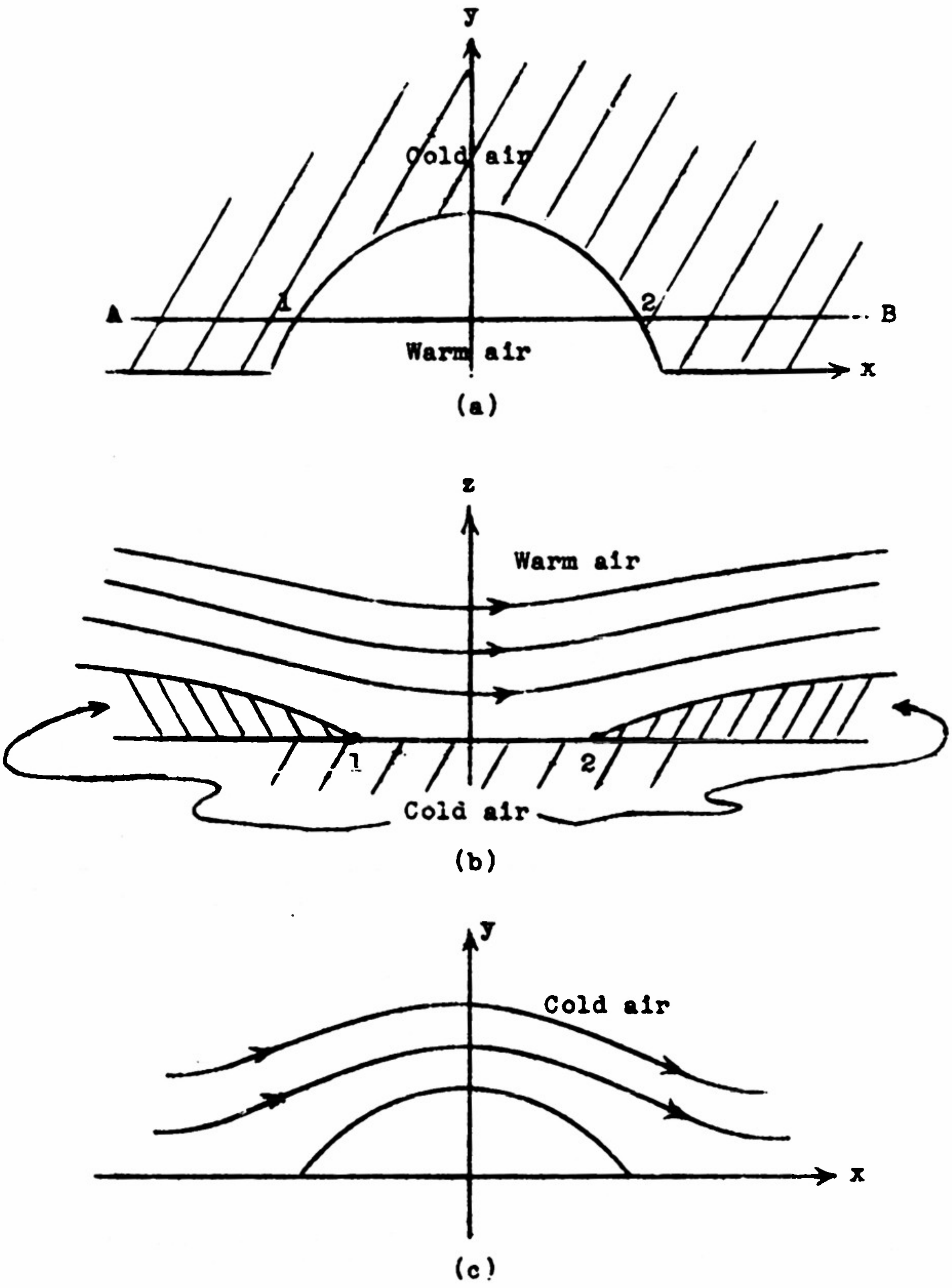

Fig. 4

quite large changes in the components u, v of the velocity in the cold air may be needed when a frontal disturbance is created. Thus we assume from now on that u', v', h' have for all time the known values they had in the initial steady state in which $v' = 0$, $u' =$ const. The differential equations for our Problem III can now be written as follows:

$$
(2.14)\quad \begin{cases} u_t + uu_x + vu_y = -g[\frac{\rho'}{\rho} h'_x + (1 - \frac{\rho'}{\rho})h_x] + \lambda v \\ v_t + uv_x + vv_y = -g[\frac{\rho'}{\rho} h'_y + (1 - \frac{\rho'}{\rho})h_y] - \lambda u \\ (uh)_x + (vh)_y + h_t = 0, \end{cases}
$$

in which h'_x and h'_y are the known functions given in terms of the initial state in the warm air. The initial state, in which $v' = v = 0$, $u =$ const., $u' =$ const., must satisfy the equations (2.9) and (2.10); this leads at once to the conditions

$$
(2.15)\quad \begin{cases} h'_y = -\frac{\lambda}{g} u', & h'_x = 0 \\ h_y = -\frac{\lambda}{g} u(1 - \frac{\rho'}{\rho})h'_y = \frac{\lambda}{g}(\frac{\rho'}{\rho} u' - u), & h_x = 0 \end{cases}
$$

for the slopes of the free surfaces initially. The slope h_y of the discontinuity surface between the two layers is nearly proportional to the velocity difference $u' - u$ since ρ'/ρ differs only slightly from unity, and it is quite small because of the factor λ, which is a fraction of the angular velocity of the earth. The differential equations for Problem III can, finally, be expressed in the form:

$$
\begin{array}{l} (2.16) \\ \text{Problem III} \end{array}\quad \begin{cases} u_t + uu_x + vu_y + g(1 - \frac{\rho'}{\rho})h_x = \lambda v \\ v_t + uv_x + vv_y + g(1 - \frac{\rho'}{\rho})h_y = \lambda(\frac{\rho'}{\rho} u' - u) \\ h_t + (uh)_x + (vh)_y = 0, \end{cases}
$$

by using the formulas for h_x' and h_y' given in (2.15). We note that the influence of the warm air expresses itself through its density ρ' and its velocity u'. The three equations (2.16) presumably have uniquely determined solutions once the values of u, v, and h are given at the initial instant $t = 0$, and such solutions might reasonably be expected to furnish an approximate description of the dynamics of frontal motions.(*) Unfortunately, these equations are still quite complicated, and they cannot be integrated numerically even with the aid of the most modern high-speed digital computers--mostly because there are three independent variables.

Consequently, one casts about for still other possibilities, either of specialization or simplification, which might yield a manageable theory. One possibility of specialization has already been mentioned: if one assumes no Coriolis force and also assumes that the motion is independent of the y-coordinate, one obtains the pair of equations

$$(2.17)\qquad \begin{cases} u_t + u u_x + g\left(1 - \dfrac{\rho'}{\rho}\right)h_x = 0 \\ h_t + (uh)_x = 0 \end{cases}$$

which are identical with the equations of the one-dimensional shallow water gravity wave theory. These equations contain in them the possibility of the development of discontinuous solutions--called bores--and this fact lies at the basis of the discussions by Freeman [3,4] and Abdullah [1]. In such one-dimensional treatments, it is clear that it is in principle not possible to deal with the bulges on fronts and their deformation in time and space, since such problems depend essentially on both space variables x and y. Another

(*)These equations are in fact quite similar to the equations for two-dimensional unsteady motion of a compressible fluid with h playing the role of the density of the fluid.

possibility would be a linearization of the differential equations (2.16) based on assuming small perturbations of the frontal surface and of the velocities from the initial uniform state. This procedure might be of some interest, since such a formulation would take care of the boundary condition at the ground, while the existing linear treatments of this problem do not. However, our interest here is in a nonlinear treatment which permits of large displacements of the fronts. One such possibility, to be discussed in the report by G. Whitham, involves essentially the integration of the first and third equations for u and h as functions of x and t, regarding y as a parameter and derivatives with respect to y as negligible and assuming initial values for v; this is feasible by the method of characteristics. Afterwards, v is determined by integrating the second equation considering u and h as known, and this can be done because the equation is a linear first order equation under these conditions. As stated earlier, this procedure furnishes good qualitative results, but it cannot be carried out in all detail numerically because of the fact that at a cold front one has, in effect, a bore propagating into a region containing no fluid and such a situation cannot be easily handled. In the following section a different approach to the problem of approximating the solution of the equations (2.16) is proposed which can be carried out numerically, but it has the disadvantage that still more physical assumptions are made and consequently the variety of motions that can be approximated is still further restricted.

Sec. 3. Problem IV: An approximate treatment of the basic problem III

The formulation of Problem IV was motivated by the following considerations. If one looks at a sequence of weather maps and thinks of the wave motion in our thin wedge of cold air, the resemblance to the motion of waves in water which deform into breakers (especially in the case of waves which develop into occluded fronts) is very strong. The great difference is that the wave motion in water takes place in the vertical plane while the wave motion in air takes place essentially in the horizontal plane. When the hydrostatic pressure assumption is made in the case of water waves the result is a theory in exact analogy to gas dynamics, and thus wave motions with an appropriate "sound speed" become possible even though the fluid is incompressible--the free surface permits the introduction of the depth of the water as a dependent quantity, this quantity plays the role of the density in gas dynamics, and thus a dynamical model in the form of a compressible fluid is obtained. Such problems have been much studied, and a good deal can be said about them, principally because the differential equations are hyperbolic and have only two independent variables. It would seem therefore reasonable to try to invent a similar theory for frontal motions in the form of a long-wave theory suitable for waves which move essentially in the horizontal, rather than the vertical, plane, and in which the waves propagate essentially parallel to the edge of the original stationary front, i.e. the x-axis. In this way one might hope to be rid of the dependence on the variable y at right angles to the stationary front, thus reducing the independent variables to two, x and t; and if one still could obtain a hyperbolic system of differential equations then numerical treatments by finite differences would be feasible. This program can, in fact, be carried out in such a way as to yield a system of four first order nonlinear differential equations in two independent and four dependent variables which are hyperbolic.

Once having decided to obtain a long-wave theory for the horizontal plane, the procedure to be followed can be inferred to a large extent from what one does in developing the same type of theory for gravity waves in water (cf. [6]). To begin with it seems clear that the displacement $\eta(x,t)$ of the front itself in the y-direction should be introduced as one of the dependent quantities--all the more since this quantity is anyway the most obvious one on the weather maps. To have such a "shallow water" theory in the horizontal plane requires--unfortunately--a rigid "bottom" somewhere (which is, of course, vertical in this case), and this we simply postulate, i.e. we assume that the y-component v of the velocity vanishes for all time on a vertical plane $y = \delta$ = const. parallel to the stationary front along the x-axis (see Figure 5). The velocity $v(x,y,t)$ is then

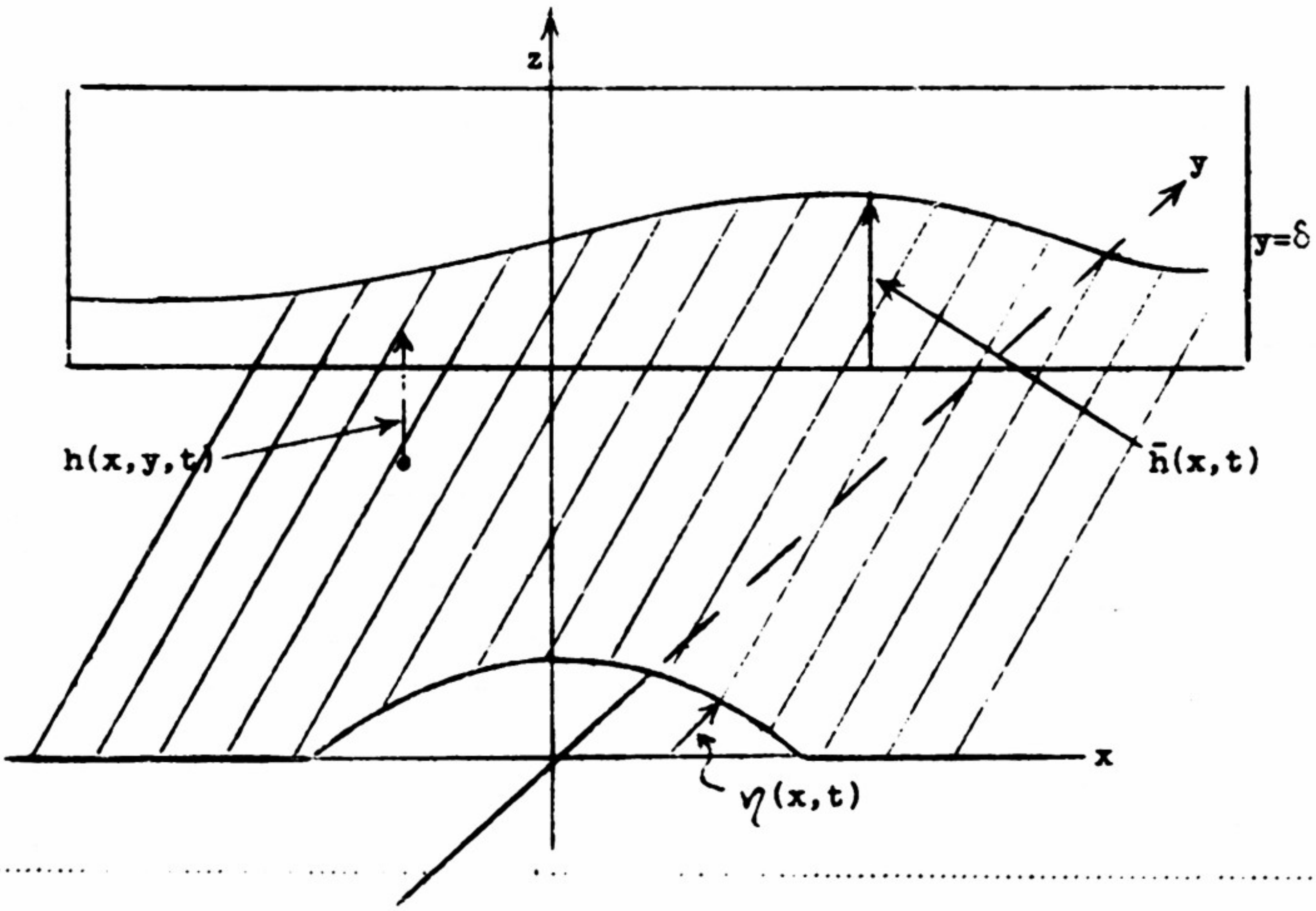

Fig. 5

assumed to vary linearly[*] in y, and its value at the front,

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[*] The analogous statement holds also in the long-wave theory in water in its simplest form.

$y = \eta(x,t)$, is called $\bar{v}(x,t)$. The intersection of the discontinuity surface $z = h(x,y,t)$ with the plane $y = \delta$ is a curve given by $z = \bar{h}(x,t)$, and we assume that the discontinuity surface is a ruled surface having straight line generators running from the front, $y = \eta(x,t)$, to the curve $z = \bar{h}(x,t)$, and parallel to the y,z-plane. Finally, we assume that u, the x-component of the velocity, depends on x and t only:(*) $u = u(x,t)$. The effect of these assumptions is to yield the relations

$$(3.1) \qquad h(x,y,t) = \frac{y - \eta(x,t)}{\delta - \eta(x,t)} \cdot \bar{h}(x,t),$$

$$(3.2) \qquad v(x,y,t) = \frac{\delta - y}{\delta - \eta(x,t)} \cdot \bar{v}(x,t),$$

as one readily sees. In addition, we assume that a particle that is once on the front $y - \eta(x,t) = 0$ always remains on it, so that the relation:

$$(3.3) \qquad \bar{v}(x,t) = \eta_t + u\eta_x$$

must hold. The four quantities $u(x,t)$, $\eta(x,t)$, $\bar{h}(x,t)$, and $\bar{v}(x,t)$ are our new dependent variables. Differential equations for them will be obtained by integrating the basic equations (2.16) of Problem III with respect to y from $y = \eta$ to $y = \delta$ --which can be done since the dependence of u, v, and h on y is now explicitly given--and these three equations together with (3.3) will yield the four equations we want.

Before writing these equations down it should be said that the most trenchant assumption is that concerning the existence of the rigid boundary $y = \delta$. One might think that as long as the velocity component v dies out with sufficient rapidity in the y-direction such an assumption would

(*) The analogous statement holds also in the long-wave theory in water in its simplest form.

yield a good approximation, but the facts in the case of water waves indicate this to be not sufficient for the accuracy of the approximation: with water waves in very deep water the vertical velocity (corresponding to our v here) dies out very rapidly in the depth, but it is nevertheless essential for a good approximation to assume that the ratio of the depth down to a rigid bottom to the wave length is small. However, such a rigid vertical barrier to the winds does exist in some cases of interest to us in the form of mountain ranges, which are often much higher than the top of the cold surface layer (i.e. higher than the curve $z = \bar{h}(x,t)$ in Figure 5). In any case, severe though this restriction is, it still seems to the author to be worth while to study the motions which are compatible with it since something about the dynamics of frontal motions with large deformations may be learned in the process. In particular, one might learn something about the kind of perturbations that are necessary to initiate motions of the type observed, and under what circumstances the motions can be maintained.

Another objection to such a theory could well be raised. At the very outset we assumed an incompressible fluid, thus ruling out at once all thermodynamical processes. This means that we consider the mechanical sources of energy to be the essential ones in the large-scale motions under study. Such a view has been advocated by Abdullah [1], who supports it by calculations showing that the over-all mechanical sources of energy are of the right order of magnitude to account for the energy in cyclones. In the end, of course, any mathematical theory which pretends to apply to frontal disturbances in the atmosphere can be justified only by its consequences in relation to the observed facts.

In carrying out the derivation of the differential equations of our theory according to the plan outlined above, we calculate first a number of integrals. The first of these arise from (3.1) and (3.2):

$$\int_{\eta}^{\delta} h\, dy = \frac{\bar{h}}{\delta - \eta} \int_{\eta}^{\delta} (y - \eta)\, dy = \frac{1}{2}\bar{h}(\delta - \eta)$$

$$\int_{\eta}^{\delta} v\, dy = \frac{\bar{v}}{\delta - \eta} \int_{\eta}^{\delta} (\delta - y)\, dy = \frac{1}{2}\bar{v}(\delta - \eta).$$

From these we derive by differentiations with respect to x and t another set of relations:

$$\int_{\eta}^{\delta} h_x\, dy = \frac{1}{2}\bar{h}_x(\delta - \eta) - \frac{1}{2}\bar{h}\eta_x,$$

$$\int_{\eta}^{\delta} v_x\, dy = \frac{1}{2}\bar{v}_x(\delta - \eta) + \frac{1}{2}\bar{v}\eta_x,$$

$$\int_{\eta}^{\delta} v_t\, dy = \frac{1}{2}\bar{v}_t(\delta - \eta) + \frac{1}{2}\bar{v}\eta_t,$$

$$\int_{\eta}^{\delta} h_t\, dy = \frac{1}{2}\bar{h}_t(\delta - \eta) - \frac{1}{2}\bar{h}\eta_t.$$

(In deriving these relations, it is necessary to observe that the lower limit η is a function of x and t.) One additional relation is needed, as follows:

$$\int_{\eta}^{\delta} (hu)_x\, dy = \frac{\partial}{\partial x}\left\{u \int_{\eta}^{\delta} h\, dy\right\}$$

$$= \frac{1}{2}(\bar{h}_x u + \bar{h}u_x)(\delta - \eta) - \frac{1}{2}\bar{h}u\eta_x.$$

We now integrate both sides of the equations (2.16) with respect to y from η to δ, make use of the above integrals, note that $u = u(x,t)$ is independent of y, and divide by $\delta - \eta$. The result is the equations

$$(3.4)\quad \begin{cases} u_t + uu_x + \frac{1}{2} k\bar{h}_x - \frac{1}{2} \frac{k\bar{h}}{\delta - \eta} \eta_x = \frac{1}{2} \lambda \bar{v}, \\ \bar{v}_t + u\bar{v}_x + \frac{u\bar{v}\,\eta_x}{\delta - \eta} + \bar{v}_t + \frac{\bar{v}}{\delta - \eta} \eta_t = \frac{\bar{v}^2}{\delta - \eta} - \frac{2k\bar{h}}{\delta - \eta} \\ \qquad - 2\lambda(u - \frac{\rho'}{\rho} u'), \\ u\bar{h}_x + \bar{h}u_x - \frac{\bar{h}u}{\delta - \eta} \eta_x + \bar{h}_t - \frac{\bar{h}}{\delta - \eta} \eta_t = 0, \end{cases}$$

with k a constant replacing the quantity $g(1 - \frac{\rho'}{\rho})$. These equations, together with (3.3), form a system of four partial differential equations for the four functions u, η, $\bar{v}$, and $\bar{h}$. By analogy with gas dynamics and the nonlinear shallow water theory, it is convenient to introduce a new dependent quantity c (which will turn out to be a "sound speed" or "propagation speed") through the relation

$$(3.5)\qquad c^2 = \frac{1}{2} k\bar{h} = \frac{1}{2} g(1 - \frac{\rho'}{\rho})\bar{h}.$$

The quantity c is real if ρ' is less than ρ, and this holds since the warm air is lighter than the cold air. In terms of this new quantity the equations (3.3) and (3.4) take the form

$$(3.6)\quad \begin{cases} u_t + uu_x + 2cc_x - \frac{c^2}{\delta - \eta} \eta_x = \frac{1}{2} \lambda \bar{v} \\ \bar{v}_t + u\bar{v}_x = - \frac{4c^2}{\delta - \eta} - 2\lambda(u - \frac{\rho'}{\rho} u') \\ 2c_t + cu_x + 2uc_x = \frac{c\bar{v}}{\delta - \eta} \\ \eta_t + u\eta_x = \bar{v}. \end{cases}$$

It is now easy to write the equations (3.6) in the characteristic form simply by replacing the first and third equations by their sum and by their difference. The result is:

$$
(3.7)\quad \begin{cases} u_t + (u+c)u_x + 2\left\{c_t + (u+c)c_x\right\} - \dfrac{c}{\delta - \eta}\left\{\eta_t + (u+c)\,\eta_x\right\} = \frac{1}{2}\,\lambda\,\bar{v}, \\ u_t + (u-c)u_x - 2\left\{c_t + (u-c)c_x\right\} + \dfrac{c}{\delta - \eta}\left\{\eta_t + (u-c)\,\eta_x\right\} = \frac{1}{2}\,\lambda\,\bar{v}, \\ \bar{v}_t + u\bar{v}_x = -\dfrac{4c^2}{\delta - \eta} - 2\lambda(u - \dfrac{\rho'}{\rho}u'), \\ \eta_t + u\eta_x = \bar{v}. \end{cases}
$$

As one sees, the equations are in characteristic form: The characteristic curves satisfy the differential equations

$$
(3.8)\qquad \frac{dx}{dt} = u + c, \quad \frac{dx}{dt} = u - c, \quad \frac{dx}{dt} = u,
$$

and each of the equations (3.7) contains only derivatives in the direction of one of these curves. The characteristic curves defined by $\frac{dx}{dt} = u$ are taken twice. Thus one sees that the quantity c is indeed entitled to be called a propagation speed, and small disturbances can be expected to propagate with this speed in both directions relative to the stream of velocity u. (In the theory by Whitham, in which the motion in each vertical plane y = const. is treated separately, the propagation or sound speed of small disturbances is given by $\sqrt{kh}$. The sound speed in the theory given here thus represents a kind of average with respect to y of the sound speeds of Whitham's theory.) Since the propagation speed depends on the height of the discontinuity surface, it is clear that the possibility of motions leading to breaking is inherent in this theory.

Once the dynamical equations have been formulated in characteristic form it becomes possible to see rather easily what sort of subsidiary initial and boundary conditions are reasonable. In fact, there are many possibilities in this respect. One such possibility, for which numerical calculations are being made, is the following. At time $t = 0$ it is assumed that u = const., $\eta = 0$, $\bar{h}$ = const. (as in a stationary front), but that $\eta_t = f(x)$ over a segment of the x-axis. In other words, it is assumed that a transverse impulse is given to the stationary front over a portion of its length. The subsequent motion is uniquely determined and can be calculated numerically. Another possibility is to prescribe a stationary front at $t = 0$ for $x > 0$, say, and then to give the values of all dependent quantities(*) at $x = 0$ as arbitrary functions of the time. One might visualize this case as one in which, for example, cold air is being either added or withdrawn at a particular point ($x = 0$ in the present case). This again yields a problem with a uniquely determined solution, and various possibilities are being explored numerically.

It was stated above that the most objectionable feature of the present theory is the assumption of a fixed vertical barrier back of the front. There is, however, a different way of looking at the problem as a whole which may mitigate this restriction somewhat. One might try to consider the motion of the entire cap of cold air that lies over the polar region, using polar coordinates (θ, φ) (with θ the latitude angle, say). One might then consider motions once more that depend essentially only on φ and t by getting rid of the dependence on θ through use of the same type of assumptions (linear behavior in θ, say) as above. Here the North Pole itself would take the place of the vertical barrier ($v = 0$!). The result would again be a system of nonlinear equations--this time with variable coefficients. Of course, it would be necessary to begin with a stationary flow in which the motion takes place along the parallels of latitude.

(*) In the numerical cases so far considered we have had $|c| < |u|$ so that even on the t-axis all four dependent quantities can be prescribed.

All in all, the ideas presented here and in the following paper of Whitham would seem to yield theories flexible enough to permit a good deal of freedom with regard to initial and other conditions so that one might hope to gain some insight into the complicated dynamics of frontal motions.